EXAM III SOLUTIONS

1. (a) $f(x) = \dfrac{1}{1+x^2} = \sum_{k=0}^{\infty}(-1)^k x^{2k}$, radius=1. The singularities of $f(x)$ are at $\pm i$, which are both one unit away from the origin.

   (b) $\tan^{-1} x = \int_0^x \dfrac{1}{1+t^2}dt = \int_0^x \sum_{k=0}^{\infty}(-1)^k t^{2k} dt = \sum_{k=0}^{\infty}(-1)^k \dfrac{x^{2k+1}}{2k+1}$, radius=1.

2. (a) The series converges on the interval $\left(-\frac{3}{2}, \frac{1}{2}\right)$, diverges when $x = \frac{1}{2}$ and converges conditionally when $x = -\frac{3}{2}$.

   (b) The series converges on the interval $\left(-\frac{2}{1+\sqrt{5}}, \frac{2}{1+\sqrt{5}}\right)$.

3. For $|x| < 1$, the geometric series $\sum_{n=0}^{\infty} x^n = \dfrac{1}{1-x}$, so for the same x, the series $\sum_{n=1}^{\infty} x^n = \dfrac{x}{1-x}$.

4. (a) converges conditionally (alternating series test, then integral test to show failure of absolute convergence)

   (b) diverges limit comparison test with $\dfrac{1}{\sqrt{n}}$ since $\lim_{n\to\infty} \dfrac{\sqrt{\sin(1/n)}}{\sqrt{1/n}} = \sqrt{\lim_{n\to\infty} \dfrac{\sin(1/n)}{1/n}} = \sqrt{\lim_{n\to\infty} \cos(1/n)} = 1$ Thus since $\dfrac{1}{\sqrt{n}}$ has a divergent series by the limit comparison test so does $\sin(1/n)$

   (c) converges (ratio test)

   (d) converges compare $\left(1+\dfrac{1}{n}\right)^n \leq \dfrac{1}{n^n} \leq \dfrac{1}{n^2}$ the last series would converge by the p-test and so the other two do by the comparison test.

5. Considering the Taylor expansion of a function $f(x)$ centered at zero $\sum_{n=0}^{\infty} \dfrac{f^{(n)}(0)}{n!}x^n$ let $f = y$ and then we see from the information given that since $\dfrac{dy}{dx} = y$ we have that $y^{(n)} = y$ and since $y(0) = 1$ we can say that $y^{(n)}(0) = 1$. Plugging this into the Taylor series expansion we have that $y(x) = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = e^x$ and this series converges for all x.