Mathematics 1210 **PRACTICE EXAM II** **Fall 2005**

ANSWER KEY

1. a. $\dfrac{dx}{dt} = A\omega\cos(\omega t - \phi)$, $\dfrac{d^2x}{dt^2} = -A\omega^2\sin(\omega t - \phi)$.

   b. $f(u) = u^3, u(x) = \dfrac{x-2}{x-\pi}$. Use chain rule

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = 3u^2\frac{(x-\pi)(x-2)' - (x-2)(x-\pi)'}{(x-\pi)^2} = 3\Big(\frac{x-2}{x-\pi}\Big)^2 . \frac{2-\pi}{(x-\pi)^2}$$

   c. Use implicit differentiation.

$$\begin{aligned} \frac{d}{dx}(\cos xy) &= \frac{d}{dx}(y^2 + 2x) \\ (-\sin xy)\frac{d}{dx}(xy) &= \frac{d}{dx}(y^2) + 2 \\ (-\sin xy)(y + x\frac{dy}{dx}) &= 2y\frac{dy}{dx} + 2 \\ \frac{dy}{dx} &= -\frac{2 + y\sin xy}{2y + x\sin xy} \end{aligned}$$

   d. $\displaystyle\lim_{x\to 0}\frac{\sin x\tan x}{1-\cos x} = \lim_{x\to 0}\frac{\sin x\sin x}{(1-\cos x)\cos x} = \lim_{x\to 0}\frac{1-\cos^2 x}{(1-\cos x)\cos x} = \lim_{x\to 0}\frac{1+\cos x}{\cos x} = 2.$

   e. $u(x) = \dfrac{\tan x}{x^2+1}, g(u) = \sqrt{u}, f(g) = \sin g$. Use chain rule

$$\begin{aligned} \frac{df}{dx} &= \frac{df}{dg}\frac{dg}{du}\frac{du}{dx} = (\cos g)\Big(\frac{1}{2\sqrt{u}}\Big)\frac{(x^2+1)(\tan x)' - \tan x(x^2+1)'}{(x^2+1)^2} \\ &= \frac{1}{2}\Big(\cos\sqrt{\frac{\tan x}{x^2+1}}\Big)\sqrt{\frac{x^2+1}{\tan x}}\,\frac{(x^2+1)\sec^2 x - 2x\tan x}{(x^2+1)^2} \end{aligned}$$

   f. Use product rule combined with the chain rule

$$\begin{aligned} \frac{df}{dx} &= \sin^2(x^3)\frac{d}{dx}(x^2) + x^2\frac{d}{dx}(\sin^2(x^3)) \\ &= \sin^2(x^3)(2x) + x^2(2\sin(x^3))\frac{d}{dx}(\sin(x^3)) \\ &= 2x\sin^2(x^3) + 2x^2\sin(x^3)\cos(x^3)(3x^2) \\ &= 2x\sin^2(x^3) + 6x^4\sin(x^3)\cos(x^3) \end{aligned}$$

2. By 1.a), the function $x(t) = A\sin(\omega t - \phi)$ has the property that $x''(t) = -A\omega^2\sin(\omega t - \phi) = -\omega^2 x(t)$, thus it is a solution for the given differential equation. Similarly we can check that $x(t) = A\cos(\omega t - \phi)$ is another solution, or we can notice that $A\cos(\omega t - \phi) = A\sin(\omega t - \phi + \pi/2)$.

   We also observe that if $x(t)$ and $y(t)$ are solutions then any linear combination of them is another solution. Indeed, if $z(t) = ux(t) + vy(t)$ with u, v are real numbers then $z''(t) = ux''(t) + vy''(t) =$

$-u\omega^2 x(t) - v\omega^2 y(t) = -\omega^2(x(t) + y(t)) = -\omega^2 z(t)$. Thus any linear combination of solutions of the form $A\sin(\omega t - \phi)$ and $A\cos(\omega t - \phi)$ gives us another solution. And these are all the possible solutions for the equation.

For $A = 2, \phi = \pi/4, \omega = 1, \quad x(t) = 2\sin(t - \pi/4)$, velocity is extremized when $x(t) = 0$, or $t = n\pi + \pi/4$, position is extremized when $v(t) = x'(t) = 0$, or $t = (n + \frac{1}{2})\pi + \pi/4$. Figure on other side.

3. Let r be the radius, $A(r)$ be the corresponding area of the enclosed oil. We have $A(r) = \pi r^2$.

   Given $\dfrac{dr}{dt} = 1.5$ (ft/sec), and the radius equals 0 when $t = 0$, then after 2 hours $r = (1.5)(2) = 3$ (in). Thus $\dfrac{dA}{dt}|_{r=3} = \pi(2r)\dfrac{dr}{dt}|_{r=3} = \pi(2)(3)(1.5) \approx 28.27(in^2)$.

4. Choose a coordinate plane so that the origin is the center of the clock, the x-axis goes through the 3:00 position, and the y-axis goes through the 12:00 position. Let $t = 0$ correspond to when the time is exactly 3:00. Let θ_1 be the angle made by the y-axis and the hour hand after t minutes, and θ_2 be the angle made by the y-axis and the minute hand after t minutes. Then the angle between the two hands after t minutes is $\theta = \theta_1 - \theta_2$.

   The hour hand completes $\dfrac{1}{12}$ of the circle after 60 minutes thus $\dfrac{d\theta_1}{dt} = \dfrac{2\pi}{720}$ (rad/min) and with a similar reason $\dfrac{d\theta_2}{dt} = \dfrac{2\pi}{60}$ (rad/min). Then $\dfrac{d\theta}{dt} = \dfrac{d\theta_1}{dt} - \dfrac{d\theta_2}{dt} = -\dfrac{11\pi}{360}$ (rad/min).
   Using the law of cosine we can find the distance between the two tips

$$d^2(t) = 4^2 + 5^2 - (2)(4)(5)\cos\theta = 41 - 40\cos\theta.$$

   To find the rate of change of the distance of the two tips at 3:00 (t=0) we use implicit differentiation

$$2d(t)d'(t) = -40(-\sin\theta)\frac{d\theta}{dt}$$

   When $t = 0, d(0) = \sqrt{4^2 + 5^2} = \sqrt{41}, \theta = \dfrac{\pi}{2}$. Thus

$$d'(0) = \frac{-11\pi}{18\sqrt{41}} = -0.299(in/min).$$

5. $f(x + \Delta x) \approx f(x) + f'(x)\Delta x$
   For $\sqrt{66}$, use $f(x) = \sqrt{x}$ with $x = 64, \Delta x = 2$. We have $f'(x) = 1/(2\sqrt{x})$. Then

$$\sqrt{66} = f(64 + 2) \approx \sqrt{64} + 1/(1\sqrt{64})(2) = 8\frac{1}{8}$$

   For $\sin\dfrac{\pi}{100}$ use $f(x) = \sin x$ with $x = 0, \Delta x = \dfrac{\pi}{100}$. We have $f'(x) = \cos x$. Then

$$\sin\frac{\pi}{100} = f(0 + \frac{\pi}{100}) \approx \sin(0) + (\cos 0)\frac{\pi}{100} = \frac{\pi}{100}.$$

#35, $m(v) = m_0\Big(1 - \frac{v^2}{c^2}\Big)^{-1/2}, \quad m'(v) = m_0\Big(-\frac{1}{2}\Big)\Big(-\frac{2v}{c^2}\Big)\Big(1 - \frac{v^2}{c^2}\Big)^{-3/2} = \frac{m_0 v}{c^2}\Big(1 - \frac{v^2}{c^2}\Big)^{-3/2}.$

Given $v = 0.9c, \Delta v = 0.02c$, we have $m(0.9c) = m_0\Big(1 - \frac{(0.9c)^2}{c^2}\Big)^{-1/2} \approx 0.29m_0$ and

$$\Delta m = m'(0.9c)(0.02c) = \frac{m_0(0.9c)}{c^2}\Big(1 - \frac{(0.9c)^2}{c^2}\Big)^{-3/2}(0.02c) \approx 0.217m_0.$$

Thus the percent increase in mass is $\frac{\Delta m}{m(0.9c)} \approx 95\%$.

6. $V = \frac{4}{3}\pi r^3$. Given $r = 3, \Delta r = 0.025$. Then

$$\Delta V \approx V'(r)\Delta r = 4\pi r^2 \Delta r = (4)(\pi)(3)^2(0.025) \approx 2.83(in^3).$$

7. a. For $x < 0, f'(x) = -1$ and for $x > 0, f'(x) = \cos x$. At $x = 0$ the function is not differentiable.
   Local maxima $(\frac{\pi}{2} + k.2\pi, 1), k \geq 0$, local minima $(0, 0)$ and $(\frac{3\pi}{2} + k.2\pi, -1), k \geq 0$.
   f is increasing on $(0, \frac{\pi}{2})$ and $(\frac{3\pi}{2} + k.2\pi, \frac{\pi}{2} + 2\pi + k.2\pi), k \geq 0$
   f is decreasing on $(\infty, 0)$ and $(\frac{\pi}{2} + k.2\pi, \frac{3\pi}{2} + k.2\pi), k \geq 0$.
   f is concave up on $(\pi + k.2\pi, 2\pi + k.2\pi 0), k \geq 0$.
   f is concave down on $(k.2\pi, \pi + k.2\pi), k \geq 0$.
   Inflection points $(k.\pi, 0), k \geq 1$.
   f does not have a global maximum, but does have global minima at $(\frac{3\pi}{2} + k.2\pi, -1), k \geq 0$

   b. $f(x) = x^3 - 12x + 1$.
   $f'(x) = 3x^2 - 12 = 0 \Longrightarrow x = 2, -2$.
   f is increasing on $(-\infty, -2)$ and $(2, \infty)$, decreasing on $(-2, 2)$.
   Local max $(-2, 17)$, local min $(2, -15)$. No global max/min.
   $f'' = 6x = 0 \Longrightarrow x = 0$.
   f is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$. Inflection point $(0, 1)$.

   c. $f(x) = \frac{1}{x^2 + 1}$.
   $f'(x) = \frac{-2x}{(x^2 + 1)^2} = 0 \Longrightarrow x = 0$.
   f is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.
   Local max (0,1). No local min. Global max (0,1). No global min.
   $f'' = \frac{6x^2 - 2}{(x^2 + 1)^3} = 0 \Longrightarrow x = \pm\frac{\sqrt{3}}{3}$.
   f is concave up on $(-\infty, \frac{-\sqrt{3}}{3})$ and $(\frac{\sqrt{3}}{3}, \infty)$. It is concave down on $(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$.
   Inflection points $(-\frac{\sqrt{3}}{3}, \frac{3}{4}), (-\frac{\sqrt{3}}{3}, \frac{3}{4})$.

8. $f(x) = 2\sin(x - \frac{\pi}{4})$, $f'(x) = 2\cos(x - \frac{\pi}{4}) = 0 \Longrightarrow x = \frac{3\pi}{4}$

   f is increasing on $(\frac{\pi}{4}, \frac{3\pi}{4})$ and decreasing on $(\frac{3\pi}{4}, \frac{5\pi}{4})$.

   Maximum value $= 2$ at $x = \frac{3\pi}{4}$. Minimum value $= 0$ at $x = \frac{5\pi}{4}$.

9. Let (x, y) be the coordinate of the right corner of the rectangle that stays on the parabola $y = 12 - x^2$. The rectangle has one side of length y, and the other of length $2x$. The area is

$$A = 2xy = 2x(12 - x^2) = 24x - 2x^3.$$

   $A' = 24 - 6x^2 = 0 \Longrightarrow x = 2$. A is maximized at $x = 2$, and the dimensions are 4×8.

10. The travel time of the light in the medium 1 is $\frac{\sqrt{x^2 + a^2}}{c_1}$, and the travel time of the light in the medium 2 is $\frac{\sqrt{(d - x)^2 + b^2}}{c_2}$. Thus the total time is

$$T(x) = \frac{\sqrt{x^2 + a^2}}{c_1} + \frac{\sqrt{(d - x)^2 + b^2}}{c_2}$$

   To find the least time we take the derivative of $T(x)$ and set it equal to 0.

$$T'(x) = \frac{x}{c_1\sqrt{x^2 + a^2}} - \frac{d - x}{c_2\sqrt{(d - x)^2 + b^2}}.$$

   Thus $T'(x) = 0$ implies that

$$\frac{x}{c_1\sqrt{x^2 + a^2}} = \frac{d - x}{c_2\sqrt{(d - x)^2 + b^2}}.$$

   Since $\sin\theta_1 = \frac{x}{\sqrt{x^2 + a^2}}$ and $\sin\theta_2 = \frac{d - x}{\sqrt{(d - x)^2 + b^2}}$ we conclude that $\frac{\sin\theta_1}{c_1} = \frac{\sin\theta_2}{c_2}$.